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Removable singularities of solutions of nonlinear totally characteristic type partial differential equations

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Abstract

Let us consider the following nonlinear singular partial differential equation $(t\partial/\partial t)^m u = F(t,x,\{(t\partial/\partial t)^j(\partial/\partial x)^\alpha u\}_{j+\alpha\le m, j<m})$ with $(t,x)\in\mathbb{C}^2$ in the complex domain. When the equation is of totally characteristic type, this equation was solved in [2] and [7] under certain Poincaré condition. In this paper, the author will discuss the removability of some kind of singularities of solutions on $\{t=0\}$.

§1. Equation and assumptions

Notations: $(t,x)\in\mathbb{C}_t\times\mathbb{C}_x$, $\mathbb{N}=\{0,1,2,\ldots\}$, and $\mathbb{N}^*=\{1,2,\ldots\}$. Let $m\in\mathbb{N}^*$, set $N=\#\{(j,\alpha)\in\mathbb{N}\times\mathbb{N}; j+\alpha\le m, j<m\}$ (that is, $N=m(m+3)/2$), and denote by $z=\{z_{j,\alpha}\}_{j+\alpha\le m, j<m}\in\mathbb{C}^N$ the complex variable in $\mathbb{C}^N$.

In this paper we will consider the following nonlinear singular partial differential equation:

$$(\mathrm{E})\qquad \left(t\frac{\partial}{\partial t}\right)^m u = F\left(t,x,\left\{\left(t\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial x}\right)^\alpha u\right\}_{\substack{j+\alpha\le m\\ j<m}}\right),$$

where $F(t,x,z)$ is a function of the variables (t,x,z) defined in a neighborhood Δ of the origin of $\mathbb{C}_t\times\mathbb{C}_x\times\mathbb{C}_z^N$, and $u=u(t,x)$ is the unknown function. Set $\Delta_0=\Delta\cap\{t=0,z=0\}$, and set also $I_m=\{(j,\alpha)\in\mathbb{N}\times\mathbb{N}; j+\alpha\le m,\ j<m\}$ and $I_m(+)=\{(j,\alpha)\in I_m; \alpha>0\}$. Let us first suppose the following conditions:

A$_1$) $F(t,x,z)$ is a holomorphic function on Δ;

A$_2$) $F(0,x,0)\equiv 0$ on Δ_0.

Then, by expanding $F(t,x,z)$ into Taylor series with respect to (t,z) we have

$$F(t,x,z)=a(x)t+\sum_{(j,\alpha)\in I_m} b_{j,\alpha}(x)z_{j,\alpha}+\sum_{p+|\nu|\ge 2} g_{p,\nu}(x)t^p z^\nu,$$

where $a(x)$, $b_{j,\alpha}(x)$ $((j,\alpha)\in I_m)$ and $g_{p,\nu}(x)$ $(p+|\nu|\geq 2)$ are all holomorphic functions on Δ_0, $\nu=\{\nu_{j,\alpha}\}_{(j,\alpha)\in I_m}\in\mathbb{N}^N$, $|\nu|=\sum_{(j,\alpha)\in I_m}\nu_{j,\alpha}$ and $z^\nu=\prod_{(j,\alpha)\in I_m}[z_{j,\alpha}]^{\nu_{j,\alpha}}$. Therefore, our equation (E) is written in the form

$$C\Big(x,t\frac{\partial}{\partial t},\frac{\partial}{\partial x}\Big)u=a(x)t+\sum_{p+|\nu|\geq 2}g_{p,\nu}(x)\,t^p\prod_{(j,\alpha)\in I_m}\Big[\Big(t\frac{\partial}{\partial t}\Big)^j\Big(\frac{\partial}{\partial x}\Big)^\alpha u\Big]^{\nu_{j,\alpha}},$$

where

$$C(x,\lambda,\xi)=\lambda^m-\sum_{(j,\alpha)\in I_m}b_{j,\alpha}(x)\lambda^j\xi^\alpha.$$

We divide our equation into the following three types:

Type (1) : $b_{j,\alpha}(x)\equiv 0$ for all $(j,\alpha)\in I_m(+)$;
Type (2) : $b_{j,\alpha}(0)\neq 0$ for some $(j,\alpha)\in I_m(+)$;
Type (3) : $b_{j,\alpha}(0)=0$ for all $(j,\alpha)\in I_m(+)$, but $b_{i,\beta}(x)\not\equiv 0$ for some $(i,\beta)\in I_m(+)$.

Type (1) is called a *Gérard-Tahara type* partial differential equation and it was studied in [3], [4] and [9]. Type (2) is called a *spacially nondegenerate type* partial differential equation and it was studied in [5]. Type (3) is called a *totally characteristic type* partial differential equation and it was studied in [2] and [7]. See also [1] and [6].

In this paper we will consider the type (3) under the following condition:

$$\text{A}_3)\qquad b_{j,\alpha}(x)=O(x^\alpha)\quad(\text{as } x\longrightarrow 0)\ \text{ for all } (j,\alpha)\in I_m(+).$$

§2. Problem in the study of singularities

By the condition A_3) we have $b_{j,\alpha}=x^\alpha c_{j,\alpha}(x)$ for some holomorphic functions $c_{j,\alpha}(x)$ $((j,\alpha)\in I_m)$. Set

$$L(\lambda,\rho)=\lambda^m-\sum_{\substack{j+\alpha\leq m\\ j<m}}c_{j,\alpha}(0)\,\lambda^j\rho(\rho-1)\cdots(\rho-\alpha+1),$$

$$L_m(X)=X^m-\sum_{\substack{j+\alpha=m\\ j<m}}c_{j,\alpha}(0)\,X^j,$$

and we denote by $c_1,\ldots,c_m$ the roots of the equation $L_m(X)=0$ in X. Then if we factorize $L(\lambda,l)$ into the form

$$L(\lambda,l)=(\lambda-\lambda_1(l))\cdots(\lambda-\lambda_m(l)),\quad l\in\mathbb{N},$$

by renumbering the subscript i of $\lambda_i(l)$ suitably we have

$$\lim_{l\to\infty}\frac{\lambda_i(l)}{l}=c_i\quad\text{for } i=1,\ldots,m.$$

On the holomorphic solution we have

Theorem 1 (Chen-Tahara [2]) *If $L(k,l) \neq 0$ holds for any $(k,l) \in \mathbb{N}^* \times \mathbb{N}$ and if $c_i \in \mathbb{C} \setminus [0,\infty)$ holds for $i = 1,\ldots,m$, the equation* (E) *has a unique holomorphic solution $u(t,x)$ in a neighborhood of $(0,0) \in \mathbb{C}_t \times \mathbb{C}_x$ satisfying $u(0,x) \equiv 0$.*

By Theorem 1 we see that in a generic case the equation (E) has one and only one holomorphic solution. This means that the other solutions of (E) must be singular if they exist. Hence, if we want to characterize the equation by the property of solutions, we need to see the structure of all the singular solutions of (E). Thus the following problem naturally arises:

Problem. *Determine all kinds of local singularities which appear in the solutions of* (E).

If we could construct all the solutions of (E) explicitly, then the problem would be solved immediately. But, in fact, it is very difficult and so it will be convenient to divide our problem into the following two parts:

(I) What kind of singularities really exists ?

(II) What kind of singularities is removable ?

(I) is the problem of the existence of singularities and (II) is the problem of non-existence of singularities.

§3. Main results

In this section we will give some results on the above problems (I) and (II). To describe the result, we will introduce the class $\widetilde{\mathcal{S}}_+$ and $\widetilde{\mathcal{O}}_+$ of functions which admit singlarities on $\{t = 0\}$.

Let us denote by:

- $\mathcal{R}(\mathbb{C} \setminus \{0\})$ the universal covering space of $\mathbb{C} \setminus \{0\}$,
- S_θ the sector $\{t \in \mathcal{R}(\mathbb{C} \setminus \{0\})\ ;\ |\arg t| < \theta\}$ in $\mathcal{R}(\mathbb{C} \setminus \{0\})$,
- $S_\theta(r)$ the domain $\{t \in S_\theta\ ;\ |t| < r\}$,
- $S(\varepsilon(s))$ the domain $\{t \in \mathcal{R}(\mathbb{C} \setminus \{0\}); 0 < |t| < \varepsilon(\arg t)\}$, where $\varepsilon(s)$ is a positive-valued continuous function on $\mathbb{R}_s$,
- D_R the disk $\{x \in \mathbb{C}\ ;\ |x| \leq R\}$,

Definition 1. (1) We denote by $\widetilde{\mathcal{S}}_+$ the set of all $u(t,x)$ satisfying the following i) and ii): i) $u(t,x)$ is a holomorphic function on $S_\theta(r) \times D_R$ for some $\theta > 0$, $r > 0$ and

$R > 0$, and ii) there is an $a > 0$ such that we have

$$\max_{x \in D_R} |u(t,x)| = O(|t|^a) \quad (\text{as } t \longrightarrow 0 \text{ in } S_\theta).$$

(2) We denote by $\widetilde{\mathcal{O}}_+$ the set of all $u(t,x)$ satisfying the following i) and ii): i) $u(t,x)$ is a holomorphic function on $S(\varepsilon(s)) \times D_R$ for some positive-valued continuous function $\varepsilon(s)$ on $\mathbb{R}_s$ and some $R > 0$, and ii) there is an $a > 0$ such that for any $\theta > 0$ we have

$$\max_{x \in D_R} |u(t,x)| = O(|t|^a) \quad (\text{as } t \longrightarrow 0 \text{ in } S_\theta).$$

From now we will adopt this class $\widetilde{S}_+$ or $\widetilde{\mathcal{O}}_+$ as a faework of solutions with singularities on $\{t = 0\}$. On the existence of singularities, we have:

Theorem 2. *Suppose the conditions:*

i) *there is a* (p,l) *such that* $\mathrm{Re}\lambda_p(l) > 0$ *and* $\lambda_p(l) \notin \mathbb{N}^*$ *hold;*

ii) *for any* $i = 1, \ldots, m$ *the convex hull of the set* $\{1, \beta, -c_i\}$ *in* $\mathbb{C}$ *does not contain the origin of* $\mathbb{C}$.

Then the equation (E) *has a solution* $u(t,x)$ *belonging in the class* $\widetilde{\mathcal{O}}_+$ *which has really singularities on* $\{t = 0\}$.

Proof. Set $\beta = \lambda_p(l)$. By the same argument as in [7] we can construct an $\widetilde{\mathcal{O}}_+$-solution $u(t,x)$ of the form

$$\begin{aligned} u(t,x) &= w(t, t(\log t), \ldots, t(\log t)^\mu, t^\beta, t^\beta(\log t), \ldots, t^\beta(\log t)^\kappa, x) \\ &= \cdots + At^\beta x^l + \cdots , \end{aligned}$$

where $w(t_0, \ldots, t_\mu, \zeta_0, \ldots, \zeta_\kappa, x)$ is a holomorphic function in a neighborhood of the origin of $\mathbb{C}_t^{1+\mu} \times \mathbb{C}_\zeta^{1+\mu} \times \mathbb{C}_x$ satisfying $w(0, \ldots, 0, x) \equiv 0$, $A \in \mathbb{C}$ is an arbitrary constant, $\mu = \#\{(i,l)\,;\, \lambda_i(l) \in \mathbb{N}^*\}$, and κ is a suitable non-negative integer satisfying $1 + \kappa \leq \#\{(i,l)\,;\, \lambda_i(l) \in S\}$ with $S = \{p + q\beta\,;\, (p,q) \in \mathbb{N} \times \mathbb{N}^*\}$. If we take $A \neq 0$, by looking at the term $At^\beta x^l$ we can conclude that this solution has really singlarities on $\{t = 0\}$. The argument of the construction is almost the same as in [7], and so we may omit the details. □

Conversely, on the non-existence of singularities we have:

Theorem 3. *Suppose the conditions:*

i) $\mathrm{Re}\lambda_i(l) \leq 0$ *for any* $l \in \mathbb{N}$ *and* $i = 1, \ldots, m$;

ii) $\mathrm{Re}\, c_i < 0$ *for* $i = 1, \ldots, m$.

If $u(t,x)$ *is a solution belonging in the class* $\widetilde{S}_+$, *then* $u(t,x)$ *is holomorphic in a neighborhood of* $(0,0) \in \mathbb{C}_t \times \mathbb{C}_x$.

Note that the condition i) implies that $\mathrm{Re}\, c_i \leq 0$ for $i = 1, \ldots, m$. Note also that in the above situation we have a unique holomorphic solution $u_0(t,x)$ satisfying $u_0(0,x) \equiv 0$. Therefore, Theorem 3 is a consequence of the following result.

Theorem 4. *Suppose the conditions:*

i) $\mathrm{Re}\lambda_i(l) \leq 0$ *for any* $l \in \mathbb{N}$ *and* $i = 1, \ldots, m$;

ii) $\mathrm{Re}\, c_i < 0$ *for* $i = 1, \ldots, m$.

Then, the uniqueness of the solution of (E) *is valid in* $\widetilde{\mathcal{S}}_+$.

§4. Sketch of the proof of Theorem 4

Suppose the conditions i) and ii) in Theorem 4. Let $u_1(t,x)$ and $u_2(t,x)$ be solutions of (E) belonging in the class $\widetilde{\mathcal{S}}_+$. Set $w(t,x) = u_1(t,x) - u_2(t,x)$. For $(q,j) \in \mathbb{N} \times \mathbb{N}$ with $q + j \leq m-1$ we set

$$\phi_{q,j}(t,\rho) = \int_0^t |L_{j+1} D_{q,j} w|(\tau, (\tau/t)^c \rho)\, \frac{d\tau}{\tau},$$

where we wrote $|f|(t,\rho) = \sum_{l \geq 0} |f_l(t)| \rho^l$ for $f(t,x) = \sum_{l \geq 0} f_l(t) x^l$,

$$\begin{aligned} L_{j+1} &= \Big(t\frac{\partial}{\partial t} - \lambda_{j+1}(\theta)\Big), \\ D_{q,j} &= (1+\theta)^{m-1-q-j}\Big(t^\mu \frac{\partial}{\partial x}\Big)^q \Theta_j, \\ \Theta_j &= \Big(t\frac{\partial}{\partial t} - \lambda_1(\theta)\Big)\Big(t\frac{\partial}{\partial t} - \lambda_2(\theta)\Big) \cdots \Big(t\frac{\partial}{\partial t} - \lambda_j(\theta)\Big), \end{aligned}$$

$\lambda_i(\theta)$ denotes the operator

$$\mathbb{C}[[x]] \ni f = \sum_{l \geq 0} f_l x^l \longmapsto \lambda_i(\theta) f = \sum_{l \geq 0} f_l \lambda_i(l) x^l \in \mathbb{C}[[x]]$$

and $(1+\theta)^{m-1-q-j} = (1 + x\partial/\partial x)^{m-1-q-j}$.

Let $\beta_0 > 0, \beta_1 > 0, \ldots, \beta_{m-1} > 0$ and set

$$\Phi(t,\rho) = \sum_{j<m} \beta_j \phi_{0,j}(t,\rho) + \sum_{\substack{q+j \leq m-1 \\ q>0}} \phi_{q,j}(t,\rho) \tag{4.1}$$

on $(0,T] \times [0,R]$. Then we have $\Phi(t,\rho) = O(t^a)$ (as $t \longrightarrow 0$) uniformly on $\rho \in [0,R]$ for some $a > 0$.

Lemma. We can find suitable $\mu > 0$, $\beta_0 > 0, \beta_1 > 0, \ldots, \beta_{m-1} > 0$, $0 < b < a$, $T_0 > 0$ and $R_0 > 0$ such that

$$t\frac{\partial}{\partial t}\Phi(t,\rho) \leq b\,\Phi(t,\rho) + M\, t^\mu \frac{\partial}{\partial \rho}\Phi(t,\rho) \tag{4.2}$$

on $(0, T_0] \times [0, R_0]$.

If we admit this lemma, then the proof of Theorem 4 is carried out in the following way.

From Lemma to Theorem 4. It is sufficient to prove that $\Phi(t,\rho) \equiv 0$ holds on $\{(t,\rho)\,;\, 0 \le t \le \varepsilon \text{ and } 0 \le \rho \le \delta\}$ for some $\varepsilon > 0$ and $\delta > 0$.

Let $b > 0$ and $M > 0$ be as in Lemma. Choose $T_1 > 0$ so that $0 < T_1 \le T_0$ and $M{T_1}^{\mu}/\mu \le R_0$ hold. Define the function $\rho(t)$ by

$$\rho(t) = M \int_t^{T_1} \frac{\tau^{\mu}}{\tau} d\tau = M({T_1}^{\mu} - t^{\mu})/\mu, \quad 0 \le t \le T_1.$$

Then, $\rho(t)$ is a solution of $t(d\rho/dt) = -Mt^{\mu}$, $0 < \rho(0) \le R_0$, $\rho(T_1) = 0$ and $\rho(t)$ is decreasing in t. Set

$$\psi(t) = t^{-b}\Phi(t,\rho(t)), \quad 0 \le t \le T_1.$$

Since $\Phi(t,\rho) = O(t^a)$ (as $t \longrightarrow 0$) uniformly on $[0,R_0]$ and since $a > b > 0$ holds, we have $\psi(t) = O(t^{a-b}) = o(1)$ (as $t \longrightarrow 0$). Moreover, by Lemma we have

$$\begin{aligned} t\frac{d}{dt}\psi(t) &= -b\,t^{-b}\Phi(t,\rho(t)) + t^{-b}\,t\frac{\partial \Phi}{\partial t}(t,\rho(t)) + t^{-b}\frac{\partial \Phi}{\partial \rho}(t,\rho(t))\,t\frac{d\rho(t)}{dt} \\ &\le -b\,t^{-b}\Phi(t,\rho(t)) + t^{-b}\Big(b\,\Phi(t,\rho(t)) + M\,t^{\mu}\frac{\partial}{\partial \rho}\Phi(t,\rho(t))\Big) \\ &\quad + t^{-b}\frac{\partial \Phi}{\partial \rho}(t,\rho(t))\,(-Mt^{\mu}) \\ &= 0 \end{aligned}$$

and therefore $(d/dt)\psi(t) \le 0$ for $0 < t \le T_1$. By integrating this from ε to $t(>0)$ we get $\psi(t) \le \psi(\varepsilon)$ for $0 < \varepsilon \le t \le T_1$ and by letting $\varepsilon \longrightarrow 0$ we have $\psi(t) \le 0$ for $0 < t \le T_1$. On the other hand, $\psi(t) \ge 0$ is clear from the definition of $\psi(t)$. Hence, we obtain $\psi(t) = 0$ for $0 < t \le T_1$: this implies

$$\Phi(t,\rho) = 0 \;\text{ on }\; \{(t,\rho)\,;\, 0 < t \le T_1 \text{ and } \rho = \rho(t)\}. \tag{4.3}$$

Since $\Phi(t,\rho)$ is increasing in ρ, (4.3) implies

$$\Phi(t,\rho) \equiv 0 \;\text{ on }\; \{(t,\rho)\,;\, 0 \le t \le T_1 \text{ and } 0 \le \rho \le \rho(t)\}.$$

This completes the proof of Theorem 4. □

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